

Braneworld PNDP and Effective Gravity

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Abstract

By desuspending a zero-dimensional manifold, one obtains Einstein manifolds with so-called “virtual dimensions.” This leads to a remarkably rich reformulation of the point particle picture, in which partially negative-dimensional product (PNDP) strings and manifolds are placed at the fore. In this paper, I explore a simple braneworld picture in which the virtual dimensions of a PNDP spacetime can be used to recover the Newtonian picture of gravity.

1 Introduction

Hidden dimensions are an intriguing and intense topic of research; for instance, in string theory, it is proposed that there are six or seven “compactified” dimensions, which wrap themselves around strings. To account for the presence of these miniscule dimensions, the string theorist must prescribe a *compactification*

$$\mathcal{M}^4 \times \mathcal{M}^H \longrightarrow \mathcal{M}^4 \times \gamma \quad (1.1)$$

where H is the number of hidden dimensions and γ is a string. When one describes the tangent manifold at γ , the result

$$T_\gamma \mathcal{M}^4 \cong \mathcal{B}^2 \quad (1.2)$$

is a two-dimensional brane. Typically, the story goes that the dimensions are encoded by some special six or seven dimensional manifold, such as a Calabi-Yau or G_2 -manifold.

However, “hidden dimensions” actually arise crucially in at least one other way: namely, we can consider performing *desuspensions* on the points of a fiber bundle. This approach is the one taken by Pigazzini, et al. [2]. In the same way that the theory of strings replaces the model of point particles, the Pigazzini theory replaces strings and D-branes by *partially negative-dimensional product manifolds*, or *PNDP manifolds*. When we take these ideas seriously, then we are left with pointlike manifolds which possess internal structure; namely, the “virtual” (or negative) dimensions of the PNDP reside *within the point itself*.

PNDPs arise naturally through *downward extension*; for instance, in bordism field theories, it is possible to “extend down to the point” if one projects a low-dimensional base manifold to a zero-dimensional total space. This technique is exhibited in [4-5]. We can carry this operation even further still by beginning with a “pointlike” manifold, endowing it with a symplectic form, and identifying it as a cross-section $\mathfrak{x}_\eta$, where $\eta = \text{int}(\mathfrak{x} \cap \mathfrak{x}_{\text{vir}}$ and $\mathfrak{x}_{\text{vir}}$ is the virtual class of the manifold X with core $\mathfrak{x}$. We can specify an equivalence relationship $x_\eta \sim *$, so that $[X/x_\eta] = [\text{PNDP}]$ is an on-the-nose class equivalence.

1.1 PNDP Basics

The most intuitive definition of a *PNDP* is as follows:

Definition 1.1. A *PNDP manifold* is a fiber bundle $J \times K$, and has dimension $\dim(J) + \dim(K)$, where $\dim(K) \leq \dim(J)$, along with a canonical projection $\pi_K : K \rightarrow J$ given by suspending K d -many times, where d is the codimension of K in J , and where $\dim(K)$ is the rank of the obstruction bundle E .

However, there is some nuance missing here. Namely, the *PNDP* is, first and foremost, a *warped product manifold* with the only abnormality being the inclusion of the virtual dimensions. In [3], a more detailed version of the *PNDP* appeared; it can be written as the product of an Einstein manifold B and a fibered manifold F :

$$PNDP = B \times_f F \quad (1.3)$$

where

$$B = \tilde{B} \times B' \quad \dim(\tilde{B}) = \tilde{n} \quad \dim(B') = n' \quad (1.4)$$

and where

$$F = \mathbb{R}^{d=n'} + E \quad \text{rank}(E) = 2 \quad (1.5)$$

such that

$$\dim(F) = -d = m \quad (1.6)$$

and there is a canonical projection $\pi_{\tilde{B}}$, where the virtual dimension of $\tilde{B}$ is equal to the dimension; these cancel out, and we are left with a single pointlike manifold, albeit with a very rich structure.

1.1.1 Constructing a PNDP

There are two ways we can cook up a *PNDP* manifold: the first is to start with a Möbius strip, and then embed that into a special point on the string's worldsheet. In this scenario, we are left with an unorientable *PNDP*. Conversely, if we construct a cylinder in the virtual portion of the *PNDP*, we are left with a zero-dimensional manifold which remarkably has the properties of a positive-dimensional one, such as orientability.

Soon after publishing the first paper, Pigazzini and colleagues made some strides towards physically applying this idea. Namely, the orientable *PNDP* has with it an associated angle:

$$\alpha = \pm \mu \beta \quad (1.7)$$

where μ is a sufficiently small quantity, and β is the phase-shift factor; in the orientable case, β is simply zero, and so the angle is trivial; however, in the Möbius-type case, β becomes π . It is conjectured that the α number can be used to mimic curvature on a flat D-brane in a way that gives rise to gravitational effects.

1.1.2 Closed and Open String Identities

Let I be an interval homeomorphic to $[0, 1]$. The open strings are defined by

$$\gamma_{open} = I_1 \times I_2 \times (I_3 + E) \quad (1.8)$$

which has an interior

$$Int_{\gamma_{open}} = I_2 \times (I_3 + E) \quad (1.9)$$

whereas the closed string has the form

$$\gamma_{closed} = S_1^1 \times S_2^1 \times (I_3 + E) \quad (1.10)$$

and its interior is written

$$Int_{\gamma_{closed}} = S_2^1 \times (I_3 + E) \quad (1.11)$$

where E is the “obstruction bundle.”

2 Gravitation

It is proposed in [3] that only the closed form of the string exhibits gravitational effects. These effects are equivalent to the transportation of a vector $\vec{v} = (v_{-d}, \dots, v_d)$ along a closed curve in $\mathcal{B}^{(d, -d)}$. The Möbius induces a *topological defect*, which becomes a $\mathbb{Z}/2\mathbb{Z}$ action when passing to the holonomy group of the string.

According to the authors of [3], the D-brane is canonically identified with $\mathbb{R}^{(d)}$, and admits a flat fundamental connection; however, due to the presence of α terms, the space will appear to have macroscopic curvature. When equipped with a Euclidean stratification, the D-brane admits an Ehressmanian decomposition into vertical and horizontal components. We can then identify these as structures in the moduli space of closed strings;

$$\mathcal{D}_{Ver}^{2n} \oplus \mathcal{D}_{Hor}^{2n} \in \mathcal{M}_{\gamma_{closed}} \quad (2.1)$$

and we obtain separate actions of α for each component, which differ by a phase β .

We introduce a line element

$$d\mathfrak{x}^2 = \mathfrak{L}_B + \mathfrak{L}_F \quad (2.2)$$

where the term $\mathfrak{L}_B$ are the Einstein contributions (“warping”), and the $\mathfrak{L}_F$ term is the fiber sector. These are set by

$$\mathfrak{L}_B = g_{B, \mu\nu} dx^\mu dx^\nu \quad (2.3)$$

and

$$\mathfrak{L}_F = \exp(\alpha) g_{F, ij} dy^i dy^j \quad (2.4)$$

and where α varies smoothly as a function of the cross-section; viz.

$$\alpha = \frac{\sum_q \mathfrak{x}_q}{\Delta t} \quad q \in \mathcal{R} \times \bar{B} \quad (2.5)$$

and $\mathcal{R}$ is the Levi-Civita field and $\bar{B} = \mathcal{D}^{2n}$ is a symplectic PNDP manifold.

We can “compress” each $\mathfrak{x}_q$ by adding virtual negative dimensions until they become pointlike. This simulates a contraction to a point. Since the “core” of $\mathfrak{x}_q$ is already a point, it will become purely virtual and therefore be hidden after the compression. If the core is a fixed point of the contraction mapping, and lies purely on either a vertical or horizontal stratum, then the cross-section will be localized entirely to that stratum. In this way, we obtain pure vertical and horizontal cross-sections. These are, respectively, artifacts of the time and space dimensions, and we can combine them together to create an idealized center of mass;

$$\mathfrak{x}_{q, Ver} \oplus \mathfrak{x}_{q, Hor} = \mathfrak{b} \quad (2.6)$$

2.1 Promoting $\mathfrak{b}$ to an Effective Datum

The $\mathfrak{b}$ element is, as stated, *idealized*. The natural urge is then to seek a map

$$\hat{f} : \mathfrak{b} \longrightarrow \hat{b} \cong \mathfrak{b}_{Eff} \quad (2.7)$$

where the image is a measurable value. Our strategy consists in four steps, and does not rely on a metric.

Firstly, we must identify the virtual class $[\mathcal{V}(\mathfrak{b})]$ and measure over it. This class represents a “space” of virtual states generated from negative dimensions, which as may be recalled, result from the confinement of $\mathfrak{x}_q$ to a single *gauge stratum* of the horizontal bundle. Let us establish a definition for the gauge strata:

Definition 2.1. *A stratum $\mathfrak{s}_0 \in \bar{B}$ is gauge if, for any groupoid $\mathcal{G} \ni \mathfrak{s}_0$, and for any gauge group $\mathcal{G} \subseteq SO(n)$, the groupoid is $\mathcal{G}$ -equivariant.*

Physically, $[\mathcal{V}(\mathfrak{b})]$ encompasses all possible states or configurations that contribute to the action of a scalar field ϕ on $\mathfrak{b}$. If we interpret this as a cohomology class, we can integrate an observable function (“test form”) over it, which transforms the virtual contributions to diffeomorphisms of the fiber component of $\bar{B}$. Specifically, let $\omega_{\mathfrak{b}} \in [\mathcal{V}(\mathfrak{b})]$ be a differential form. Then, roughly speaking:

$$\int_{[\mathcal{V}(\mathfrak{b})]} \omega_{\mathfrak{b}} = S_{Eff} \quad (2.8)$$

where S is the Polyakov action.

Next, we introduce an observable function that couples to the macroscopic curvature induced by $\mathfrak{b} \times \gamma_{closed}$. This reads as a scalar field $\phi(\mathfrak{b})$ which depends on the curvature R :

$$\phi(\mathfrak{b}) = \int_{[\mathcal{V}(\mathfrak{b})]} R \omega_{\mathfrak{b}} = S_{R, Eff} \quad (2.9)$$

Since $\mathfrak{b}$ lies at the intersection of base and fiber sectors, it obtains contributions from both $\mathfrak{L}_B$ and $\mathfrak{L}_F$; we extend this coupling to observable quantities in both sectors.

Example 2.1. *Define two effective masses (or energy densities) m_B and m_F that depend on integrals over the virtual class; viz.*

$$m_B = \int_{[\mathcal{V}(\mathfrak{b})]} \mathfrak{L}_B \quad m_F = \int_{[\mathcal{V}(\mathfrak{b})]} \mathfrak{L}_F \quad (2.10)$$

For instance, m_B could correspond to the presence of an observable gravitational field in the base sector, whereas m_F might induce torsion or affect fiber structure.

Finally, we set

$$\int_{\bar{B}} \mathfrak{b}_q = \phi(\mathfrak{b}) \frac{m_{q,B} m_{q',F}}{(d(q, q'))^2} = \hat{b} \quad (2.11)$$

which recovers the Newtonian action at finite distances.

A References

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